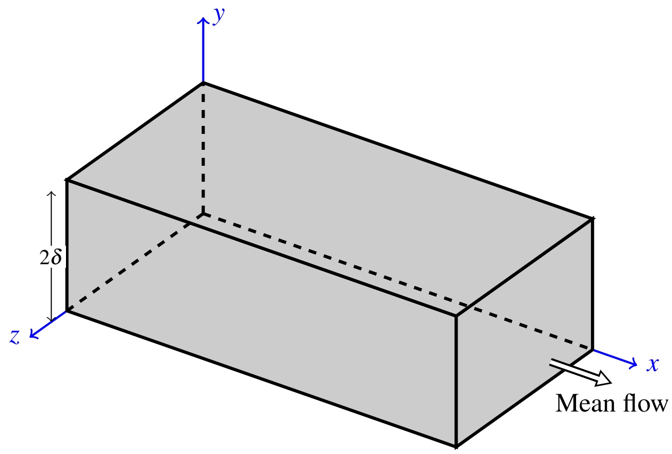
# Definitions

- $U_1$, streamwise mean velocity ($U_2 \equiv 0$; $U_3 \approx 0$)
- $u_1, u_2, u_3$, are the fluctuating velocities along $x$, $y$, $z$, coordinates respectively
- $\nu$, kinematic viscosity
- Homogeneous along $x$ and $z$ direction, i.e., $\partial_x\overline{(\cdot)} = \partial_z\overline{(\cdot)} = 0$




# Statistical moments and their budgets

The second-, third- and fourth-order moments of the Navier-Stokes equations for the case of parallel flow are given in Equations (1), (2) and (3), respectively. The terms written on the right-hand-side balance the total derivative. For parallel-flows the spatial gradient in the convective term is zero. Furthermore, with a constant streamwise pressure gradient, the temporal derivative is zero. Hence in the unstrained fully-developed channel flow, $D_t = 0$. Due to straining the material derivative is $\mathcal{D}_t = \partial_t \neq 0$. The applied strain causes the moments to evolve in time, anologous to spatial development of an adverse pressure gradient boundary layer. In otherwords, spatial derivative is replaced by a time derivative.

$$\partial_t\overline{u_iu_j} = \mathcal{P}^S_{ij} + \mathcal{P}^A_{ij} + \mathcal{T}_{ij} + \Pi_{ij} + \varepsilon_{ij} + \mathcal{D}_{ij} \tag{1}$$

where

$$\begin{aligned}
\mathcal{P}^S_{ij} &= -\overline{u_iu_2}\partial_2U_j - \overline{u_ju_2}\partial_2U_i\\
\mathcal{P}^A_{ij} &= -\overline{u_iu_k}A_{jk} - \overline{u_ju_k}A_{ik}\\
\mathcal{T}_{ij} &= -\partial_2\overline{u_2u_iu_j}\\
\Pi_{ij} &= -(1/\rho)(\overline{u_j\partial_ip} + \overline{u_i\partial_jp})\\
\varepsilon_{ij} &= -2\nu\overline{\partial_ku_i\partial_ku_j}\\
\mathcal{D}_{ij} &= \nu\partial_2^2\overline{u_iu_j}
\end{aligned}$$

$$\partial_t\overline{u_iu_ju_l} = \mathcal{P}^S_{ijl} + \mathcal{P}^A_{ijl} + \mathcal{P}^T_{ijl} + \mathcal{T}_{ijl} + \Pi_{ijl} + \varepsilon_{ijl} + \mathcal{D}_{ijl} \tag{2}$$

where

$$\begin{aligned}
\mathcal{P}^S_{ijl} &= -\overline{u_iu_ju_2}\partial_2U_l - \overline{u_ju_lu_2}\partial_2U_i - \overline{u_lu_iu_2}\partial_2U_j\\
\mathcal{P}^A_{ijl} &= -\overline{u_iu_ju_k}A_{lk} - \overline{u_ju_lu_k}A_{ik} - \overline{u_lu_iu_k}A_{jk}\\
\mathcal{P}^T_{ijl} &= \overline{u_iu_j}\partial_2\overline{u_2u_l} + \overline{u_ju_l}\partial_2\overline{u_2u_i} + \overline{u_lu_i}\partial_2\overline{u_2u_j}\\
\mathcal{T}_{ijl} &= -\partial_2\overline{u_2u_iu_ju_l}\\
\Pi_{ijl} &= -(1/\rho)(\overline{u_iu_j\partial_lp} + \overline{u_ju_l\partial_ip} + \overline{u_lu_i\partial_jp})\\
\varepsilon_{ijl} &= -2\nu(\overline{u_i\partial_ku_j\partial_ku_l} + \overline{u_j\partial_ku_i\partial_ku_l} + \overline{u_l\partial_ku_i\partial_ku_j})\\
\mathcal{D}_{ijl} &= \nu\partial_2^2\overline{u_iu_ju_l}
\end{aligned}$$

$$\partial_t \overline{u_i u_j u_l u_m} = \mathcal{P}^S_{ijlm} + \mathcal{P}^A_{ijlm} + \mathcal{P}^T_{ijlm} + \mathcal{T}_{ijlm} + \Pi_{ijlm} + \varepsilon_{ijlm} + \mathcal{D}_{ijlm} \tag{3}$$

where

$$\begin{aligned}
\mathcal{P}^S_{ijlm} &= -\overline{u_i u_j u_l u_2}\partial_2 U_m - \overline{u_j u_l u_m u_2}\partial_2 U_i - \overline{u_l u_m u_i u_2}\partial_2 U_j - \overline{u_m u_i u_j u_2}\partial_2 U_l \\
\mathcal{P}^A_{ijlm} &= -\overline{u_i u_j u_l u_k} A_{mk} - \overline{u_j u_l u_m u_k} A_{ik} - \overline{u_l u_m u_i u_k} A_{jk} - \overline{u_m u_i u_j u_k} A_{lk} \\
\mathcal{P}^T_{ijlm} &= \overline{u_i u_j u_l}\partial_2 \overline{u_2 u_m} + \overline{u_j u_l u_m}\partial_2 \overline{u_2 u_i} + \overline{u_l u_m u_i}\partial_2 \overline{u_2 u_j} + \overline{u_m u_i u_j}\partial_2 \overline{u_2 u_l} \\
\mathcal{T}_{ijlm} &= -\partial_2 \overline{u_2 u_i u_j u_l u_m} \\
\Pi_{ijlm} &= -(1/\rho)(\overline{u_i u_j u_l \partial_m p} + \overline{u_j u_l u_m \partial_i p} + \overline{u_l u_m u_i \partial_j p} + \overline{u_m u_i u_j \partial_l p}) \\
\varepsilon_{ijlm} &= -2\nu(\overline{u_i u_j \partial_k u_l \partial_k u_m} + \overline{u_i u_l \partial_k u_j \partial_k u_m} + \overline{u_j u_l \partial_k u_i \partial_k u_m} \\
&\quad + \overline{u_j u_m \partial_k u_l \partial_k u_i} + \overline{u_l u_m \partial_k u_j \partial_k u_i} + \overline{u_m u_i \partial_k u_l \partial_k u_j}) \\
\mathcal{D}_{ijlm} &= \nu \partial_2^2 \overline{u_i u_j u_l}
\end{aligned}$$

In equation 1-3, budgets on the right hand side are termed as 'production' due to shear ($\mathcal{P}^S$), 'production' due to Reynolds stress ($\mathcal{P}^T$), turbulent transport ($\mathcal{T}$), velocity-pressure gradient correlation ($\Pi$), 'dissipation' ($\varepsilon$) and viscous diffusion ($\mathcal{D}$). The applied strain produces source terms in all the moment equations, which is called 'production' due to applied strain ($\mathcal{P}^A_{ij}$). The terminologies used to describe the budget terms are inherited from second-moment closure, and do not necessarily represent physical behavior (e.g., 'production' can be negative and 'dissipation' can be positive).

The velocity pressure gradient term is decomposed as $\Pi = \psi + \phi$, where $\psi$ is pressure transport term and $\phi$ the pressure-strain term. Expressions of the decomposed terms in the second-, third- and fourth-order moment are given below.

$$\psi_{ij} = \frac{-1}{\rho}\left(\partial_2 \overline{p u_j}\delta_{2i} + \partial_2 \overline{p u_i}\delta_{2j}\right) \tag{4}$$

$$\phi_{ij} = \frac{1}{\rho}\left(\overline{p \partial_i u_j} + \overline{p \partial_j u_i}\right) \tag{5}$$

$$\psi_{ijl} = \frac{-1}{\rho}\left(\partial_2 \overline{p u_i u_j}\delta_{2l} + \partial_2 \overline{p u_j u_l}\delta_{2i} + \partial_2 \overline{p u_l u_i}\delta_{2j}\right) \tag{6}$$

$$\phi_{ijl} = \frac{1}{\rho}\left(\overline{p \partial_l u_i u_j} + \overline{p \partial_i u_j u_l} + \overline{p \partial_j u_l u_i}\right) \tag{7}$$

$$\psi_{ijlm} = \frac{-1}{\rho}\left(\partial_2 \overline{p u_i u_j u_l}\delta_{2m} + \partial_2 \overline{p u_j u_l u_m}\delta_{2i} + \partial_2 \overline{p u_l u_m u_i}\delta_{2j} + \partial_2 \overline{p u_m u_i u_j}\partial_{2l}\right) \tag{8}$$

$$\phi_{ijlm} = \frac{1}{\rho}\left(\overline{p \partial_m u_i u_j u_l} + \overline{p \partial_i u_j u_l u_m} + \overline{p \partial_j u_l u_m u_i} + \overline{p \partial_l u_m u_i u_j}\right) \tag{9}$$