Summary of Uncertainty Procedure

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The basic uncertainty estimation procedure from the Fluids Engineering Division of the ASME [1] is employed, along with some minor variations. Three representative grid sizes, h_i, are obtained from the three finest grids. This is done using:

$$h_i = \left(\frac{1}{N_i}\right)^A \tag{1}$$

where N_i is the number of unknowns (grid size) for the ith grid and $A = 1/2$ for 2-D and $1/3$ for 3-D. The grids should be in the same "family." For example, in a structured-grid family, each successively coarser grid is formed by taking every other grid point in each coordinate direction from the next finer grid. The grid ratios are defined as:

$$r_{21} \equiv h_2/h_1 \qquad r_{32} \equiv h_3/h_2 \tag{2}$$

where h_1 represents the finest of the three grids, and h_3 the coarsest. Then, with ϕ_1, ϕ_2, and ϕ_3 representing the corresponding three solutions on each grid, the solution differences are defined as:

$$\varepsilon_{21} \equiv \phi_2 - \phi_1 \qquad \varepsilon_{32} \equiv \phi_3 - \phi_2 \tag{3}$$

The apparent order p is found using fixed point iteration from the following:

$$p = \frac{1}{ln(r_{21})}(ln|\varepsilon_{32}/\varepsilon_{21}| + q(p)) \tag{4}$$

$$q(p) = ln\left(\frac{r_{21}^p - s}{r_{32}^p - s}\right) \tag{5}$$

$$s = 1 \times sign(\varepsilon_{32}/\varepsilon_{21}) \tag{6}$$

Note that if $\varepsilon_{32}/\varepsilon_{21} \leq 0$ then the convergence is "oscillatory" (non-monotonic). Also note that the above expression for p above is different than the expression in Ref. [1], in that the absolute value is *not* taken of the quantity $(ln|\varepsilon_{32}/\varepsilon_{21}| + q(p))$. This is because we want to be able to recognize when the apparent computed order is non-positive (divergent).

The approximate relative fine-grid error is:

$$e_a^{21} = \left|\frac{\phi_1 - \phi_2}{\phi_1}\right| \tag{7}$$

The extrapolated relative fine-grid error is:

$$e_{ext}^{21} = \left|\frac{\phi_{ext}^{21} - \phi_1}{\phi_{ext}^{21}}\right| \tag{8}$$

where ϕ_{ext}^{21} is the extrapolated value of the solution using:

$$\phi_{ext}^{21} = (r_{21}^p\phi_1 - \phi_2)/(r_{21}^p - 1) \tag{9}$$

The basic fine-grid convergence index, GCI_{fine}^{21}, is computed from:

$$GCI_{fine}^{21} = \frac{1.25e_a^{21}}{r_{21}^p - 1} \tag{10}$$

where the 1.25 in the expression is the recommended "safety factor." The GCI^{21}_{fine} is expressed in $\%$ by multiplying it by 100. The solution itself can be expressed as the fine grid value plus or minus its uncertainty based on GCI^{21}_{fine}:

$$\phi \approx \phi_1 \pm (GCI^{21}_{fine})|\phi_1| \tag{11}$$

Further refinements to GCI^{21}_{fine} have been made for many of the later results posted to the TMR website, based on ideas from Eça and Hoekstra [2]. When the computed apparent order p is positive, but below some cutoff value C_{low} (taken here as $C_{low} = 0.95$), the GCI^{21}_{fine} is limited based on a factor of the maximum difference (in absolute value) between any of the solutions, $\Delta_M = max(|\phi_2 - \phi_1|, |\phi_3 - \phi_2|, |\phi_3 - \phi_1|)$:

$$GCI^{21}_{fine} = min(\frac{1.25e^{21}_a}{r^p_{21} - 1}, 1.25\Delta_M/|\phi_1|) \tag{12}$$

When the computed apparent order p is above some cutoff value C_{hi} (taken here as $C_{hi} = 3.05$), then an apparent order of $p \approx C_{hi}$ is imposed, and again GCI^{21}_{fine} is limited based on a factor of Δ_M:

$$GCI^{21}_{fine} = max(\frac{1.25e^{21}_a}{r^3_{21} - 1}, 1.25\Delta_M/|\phi_1|) \tag{13}$$

For cases with oscillatory convergence ($\varepsilon_{32}/\varepsilon_{21} \leq 0$) or for cases with non-positive apparent order ($p \leq 0$), then determination of GCI^{21}_{fine} is more difficult. For the TMR website, no value is given ("N/A"). However, Ref. [2] suggests the following:

$$GCI^{21}_{fine} = 3\Delta_M/|\phi_1| \tag{14}$$

for non-monotonic convergence.

References

[1] Celik, I. B., Ghia, U., Roache, P. J., Freitas, C. J., Coleman, H., Raad, P. E., "Procedure for Estimation and Reporting of Uncertainty Due to Discretization in CFD Applications," *Journal of Fluids Engineering*, Vol. 130, July 2008, 078001.

[2] Eça, L. and Hoekstra, M., "Evaluation of Numerical Error Estimation Based on Grid Refinement Studies with the Method of Manufactured Solutions," *Computers and Fluids*, Vol. 38, 2009, pp. 1580–1591.