Turbulence Modeling: Roadblocks,
and the Potential for Machine Learning
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Conjectures of a Generalized Law of the Wall and a Structural Limitation for Classical Turbulence Models

Philippe Spalart
Retired, NASA/Boeing

Helpful comments from Strelets, Rumsey, and Batten

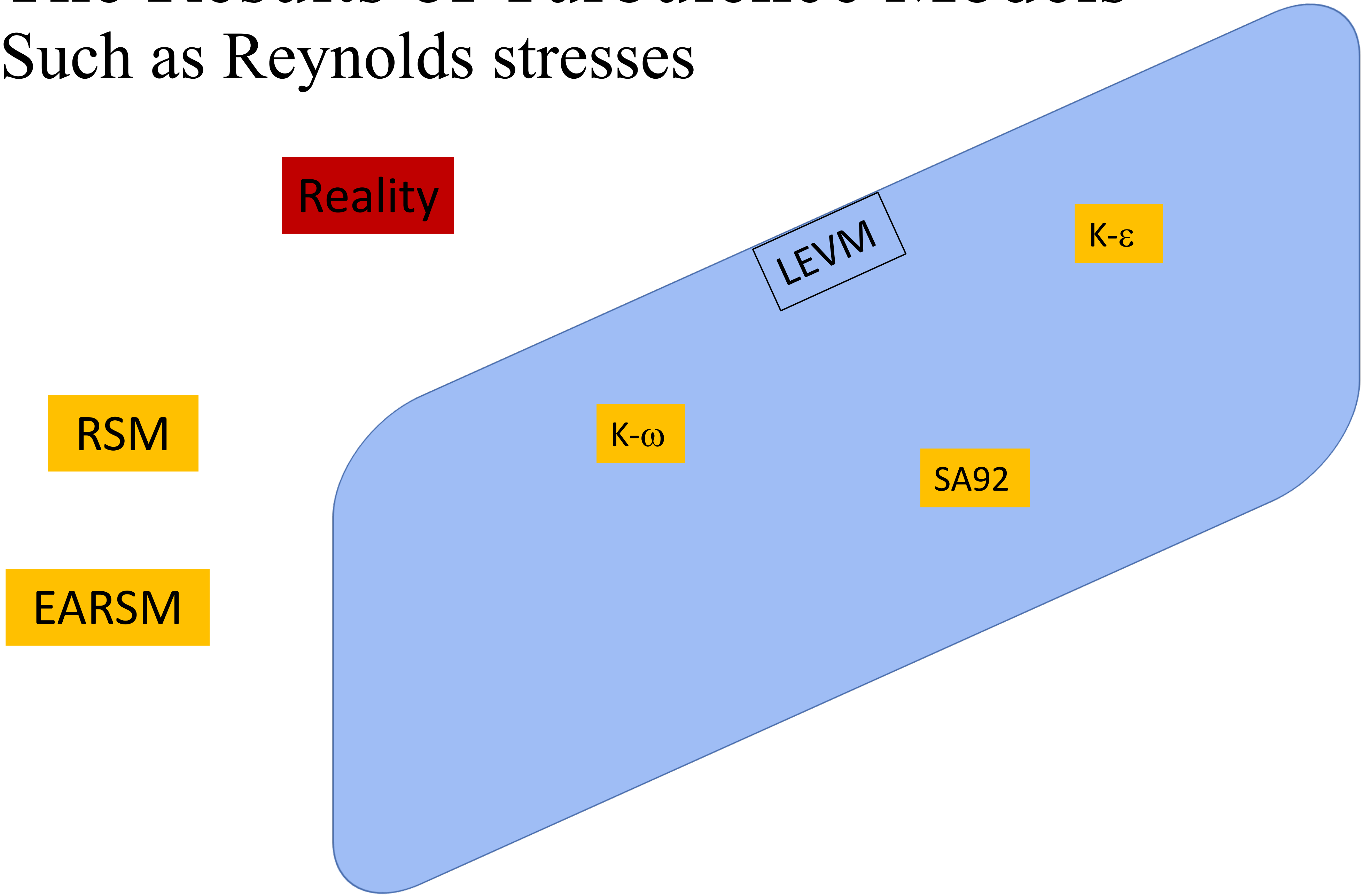
Outline

- We define the class of "Classical RANS Turbulence Models"
  - Example: k-ε
- We formulate a conjecture we call Generalized Law of the Wall about any quantity Q in predictions of the constant-stress layer:

$$Q = f(y^+)\, C\, u_\tau^\alpha\, y^\beta, \qquad \lim_{y^+\to\infty} f(y^+) = 1$$

  - It is not really new, but is more specific here, and extends to the wall
  - We don't have a mathematical proof, but we have new arguments
  - Analytical and numerical results support it
- If true, it prevents **any** Classical Model from fully emulating Reality, even for the Reynolds stresses in channel flow
  - The implications for our paradigm in Machine Learning are evident

The Results of Turbulence Models
Such as Reynolds stresses



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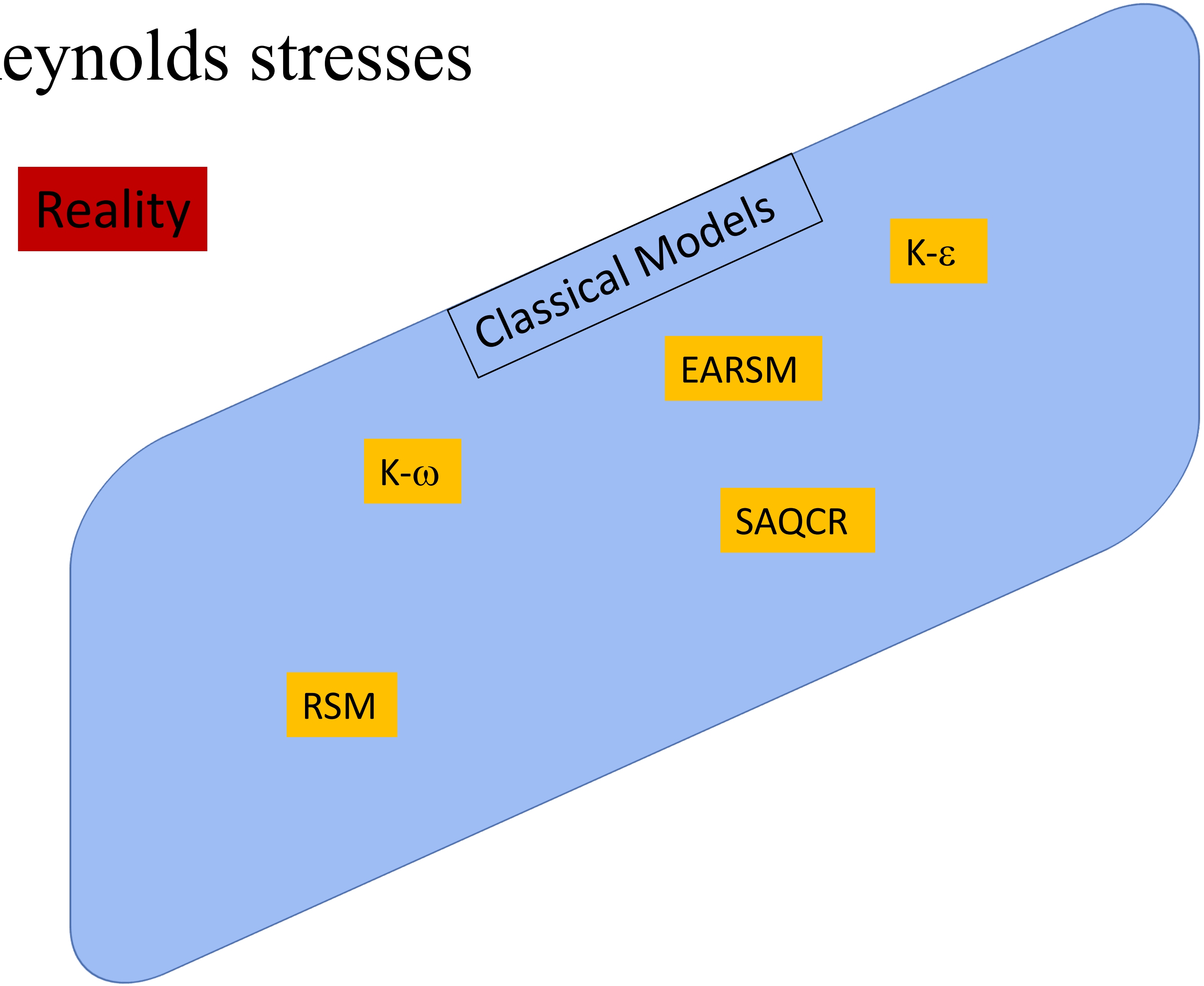

Classical Turbulence Models

- Transport models consist in (1) Constitutive Relations, (2) Budgets, and (3) Viscous Functions
- (1) The Constitutive Relation (if an eddy viscosity) is a combination of the turbulence quantities, Q_i, and the velocity-gradient tensor
- (2) The Evolution Equation, or Budget, for each quantity Q_i, includes:
  - A Production Term proportional to the velocity-gradient tensor
  - Internal Source terms
  - Internal Diffusion terms
  - Possibly terms involving the wall distance
- The first three imitate the exact Reynolds-Stress transport equation
  - Although the models have much empirical content
    - Wall distance is an arbitrary addition
  - Of course, the "sacred" RS equation is not closed
- I believe the GLW is an unintended consequence of this "imitation"
  - Together with the strong demand from CFD for locality
- (3) The viscous functions are active only up to the beginning of the log layer
  - After that, $f(.) = 1$

The Constant-Stress Layer

- The qualitative reasoning is general, but the math is written in a constant-stress wall-bounded layer
  - The total shear stress is $\tau = u_\tau^2$, setting $\rho = 1$
- The velocity Law of the Wall is accepted:

$$U = u_\tau \, U^+(y^+)$$

- As is the logarithmic law:

$$U^+ = \frac{1}{\kappa}\log(y^+) + C, \qquad\qquad y^+ \to \infty$$

- y is much smaller than the thickness of the flow: $y \ll h$
- h^+ is very large, and results "improve" as it increases

The Velocity and the Generalized Laws of the Wall

- The shear rate satisfies $$\frac{dU}{dy} = \left(y^+ \ \frac{dU^+}{dy^+}\right) \frac{u_\tau}{\kappa y}$$

and $\frac{y^+ dU^+}{dy^+}$ is a function of y^+ only; we call it $f_{dU/dy}(y^+)$

$$\lim_{y^+ \to \infty} f_{dU/dy} = 1$$

- The GLW *conjecture* states that similarly for any quantity Q in the model,

$$Q = f_Q(y^+)\, C_Q\, u_\tau^{\alpha_Q}\, y^{\beta_Q}, \qquad \lim_{y^+ \to \infty} f_Q(y^+) = 1$$

  - Here, α_Q and β_Q are dictated by dimensional analysis
  - For instance, $C_{dU/dy} = 1/\kappa$, $\alpha_{dU/dy} = 1$, $\beta_{dU/dy} = -1$
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Arguments in Favor of the GLW in Model Results

- We examine the budgets. They contain:
  - Algebraic combinations of the Q's such as $k\omega$ and $Re_t \equiv \frac{k^2}{\epsilon\nu}$ and $a_{ij} \equiv \frac{R_{ij}}{k} - \frac{2}{3}\delta_{ij}$
  - Derivatives such as $d\omega/dy$
  - Non-dimensional functions such as $f_1(Re_t)$
- All of these terms satisfy the GLW, provided the Q quantities do:
  - The algebraic combinations inherit the dimensions and the $f_Q(y^+)$ dependencies
  - $\frac{dQ}{dy} = \left(y^+ f_Q' + \beta_Q f_Q\right) C_Q\, u_\tau^{\alpha_Q} y^{\beta_Q - 1}$, and therefore obeys the GLW
  - Re_t and χ don't satisfy the GLW, because $\lim_{y^+ \to \infty} Re_t = \infty$, but the functions $f_1(Re_t)$ and $f_{v1}(\chi)$ do
- As a result, the entire budget DQ/Dt obeys the GLW
  - In the log layer, where all the f's equal 1, it contains the C_Q's and the model
  - We have N+1 equations for N+1 unknowns, namely κ and the C_Q's
  - We find κ and the C_Q's such that $C_B = 0$, and then f_B does not matter
  - The f_Q functions depend on the viscous functions of the model, such as f_{v1} or f_1, and the actual viscous terms

Example: the Chien k-ε Model

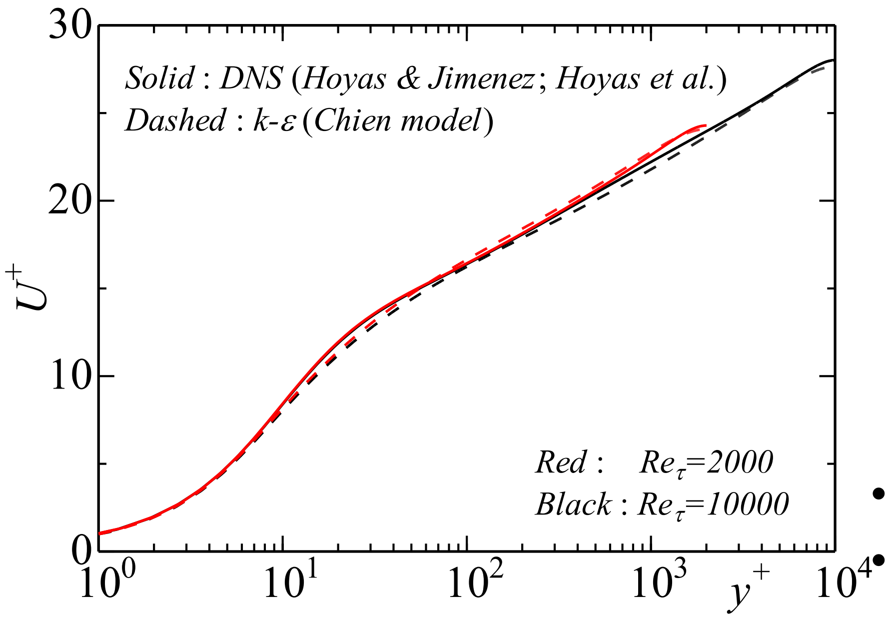
- We work in the inviscid region, where all the f functions equal 1
- Capital C's will be as in the GLW, while lower-case c's are the constants of the model, e.g. c_μ
- We assume that $$\frac{dU}{dy} = u_\tau / \kappa y$$
- There are three unknowns, namely κ, C_k, and C_ε, and three equations:

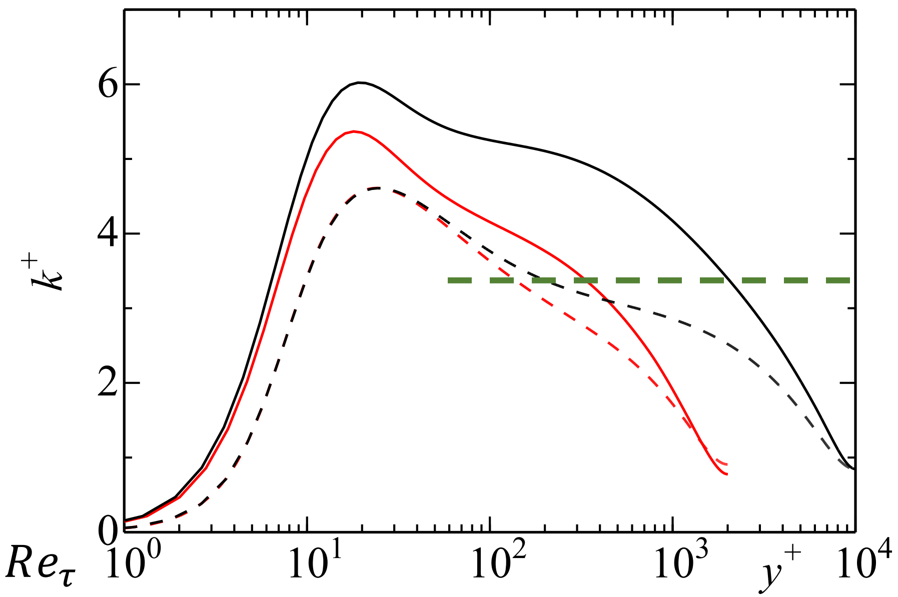
$$\tau - \tau_{wall} = u_\tau^2 \left(\frac{c_\mu C_k^2}{\kappa C_\epsilon} - 1 \right) = 0,$$

$$\frac{Dk}{Dt} = \frac{u_\tau^3}{y} \left(\frac{1}{\kappa} - C_\epsilon \right) = 0, \qquad \frac{D\epsilon}{Dt} = \frac{u_\tau^4}{y^2} \left(\frac{c_{\epsilon 1}}{\kappa} \frac{C_\epsilon}{C_k} - c_{\epsilon 2} \frac{C_\epsilon^2}{C_k} + \frac{2\kappa}{\sigma_\epsilon} \right) = 0$$

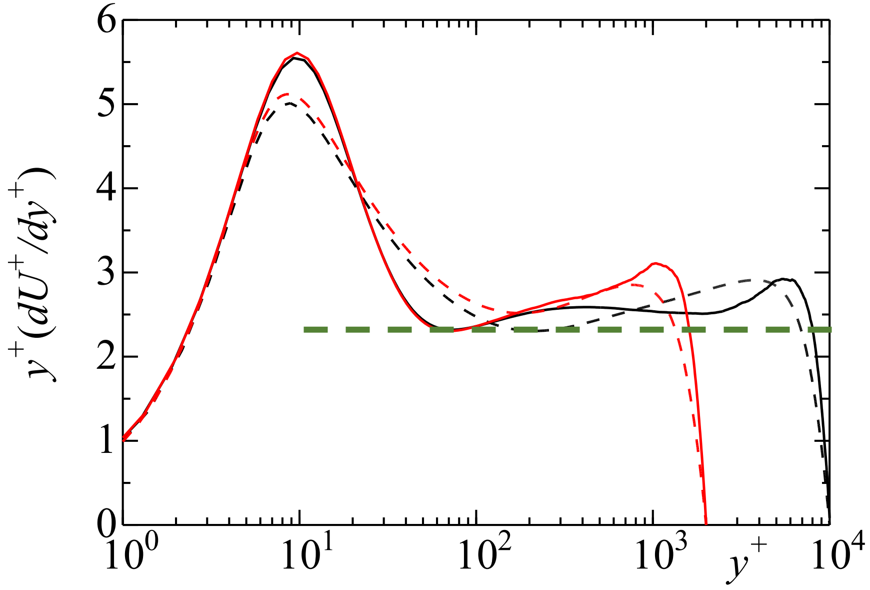
- These equations accept a solution in which C_k and C_ϵ are constants,
  - and independent of the flow type (channel, boundary layer) and Reynolds number
  - This is not a proof that this solution is unique, for other models
- The first two combined give $c_\mu C_k^2 = 1$, which is the well-known $k^+ = 1/\sqrt{c_\mu}$, and $C_{dU/dy} = C_\epsilon = 1/\kappa$, i.e., the accepted behavior
- The third equation sets *κ=0.444!*

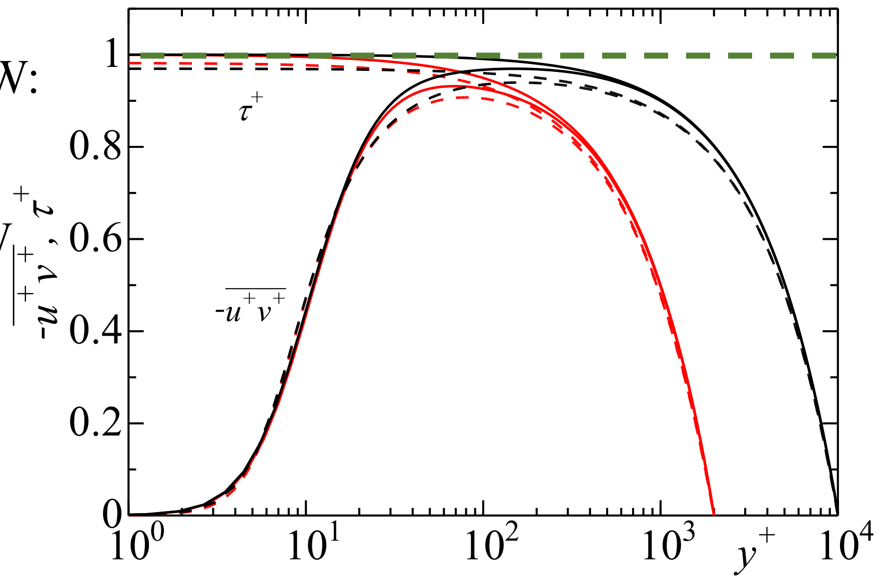
Chien $k - \epsilon$ Model in Channel Flow





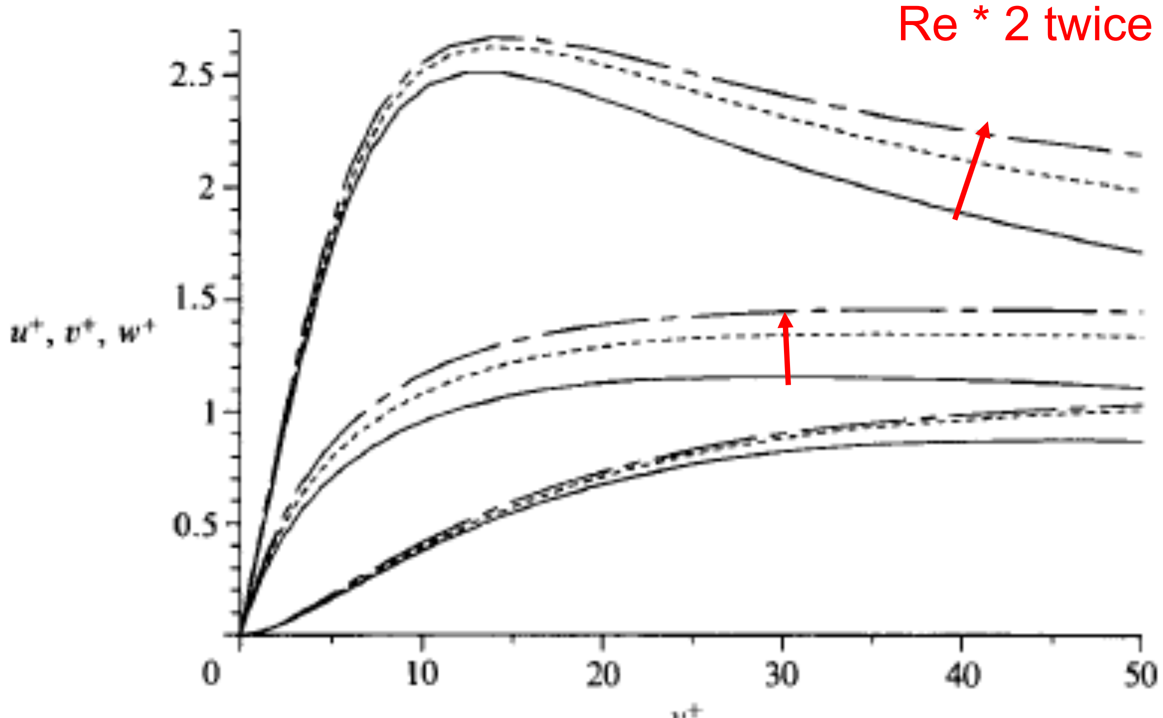
- Figures by H. Abe
- τ^+ is slowly approaching 1 at higher Re_τ
- *dU/dy* from DNS and model in agreement with LW
  - Considering statistical scatter
- Model results for *k* approach the GLW:
  - Hint of constant behavior in log layer
  - No dependence on Re_τ (“$f(y^+)$”)
- DNS results for *k* invalidate the GLW
- $Re_\tau = 10^4$ is insufficient

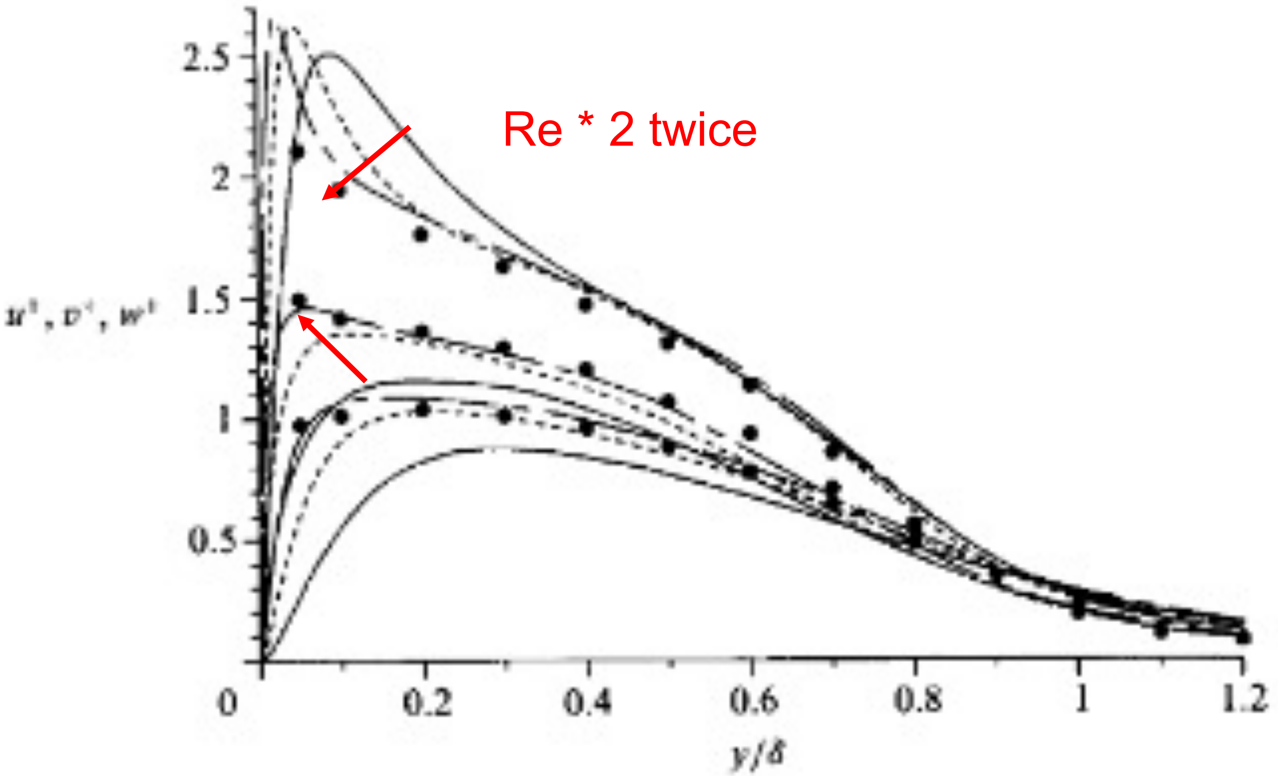




Reynolds Stresses in Boundary-Layer DNS, 1988

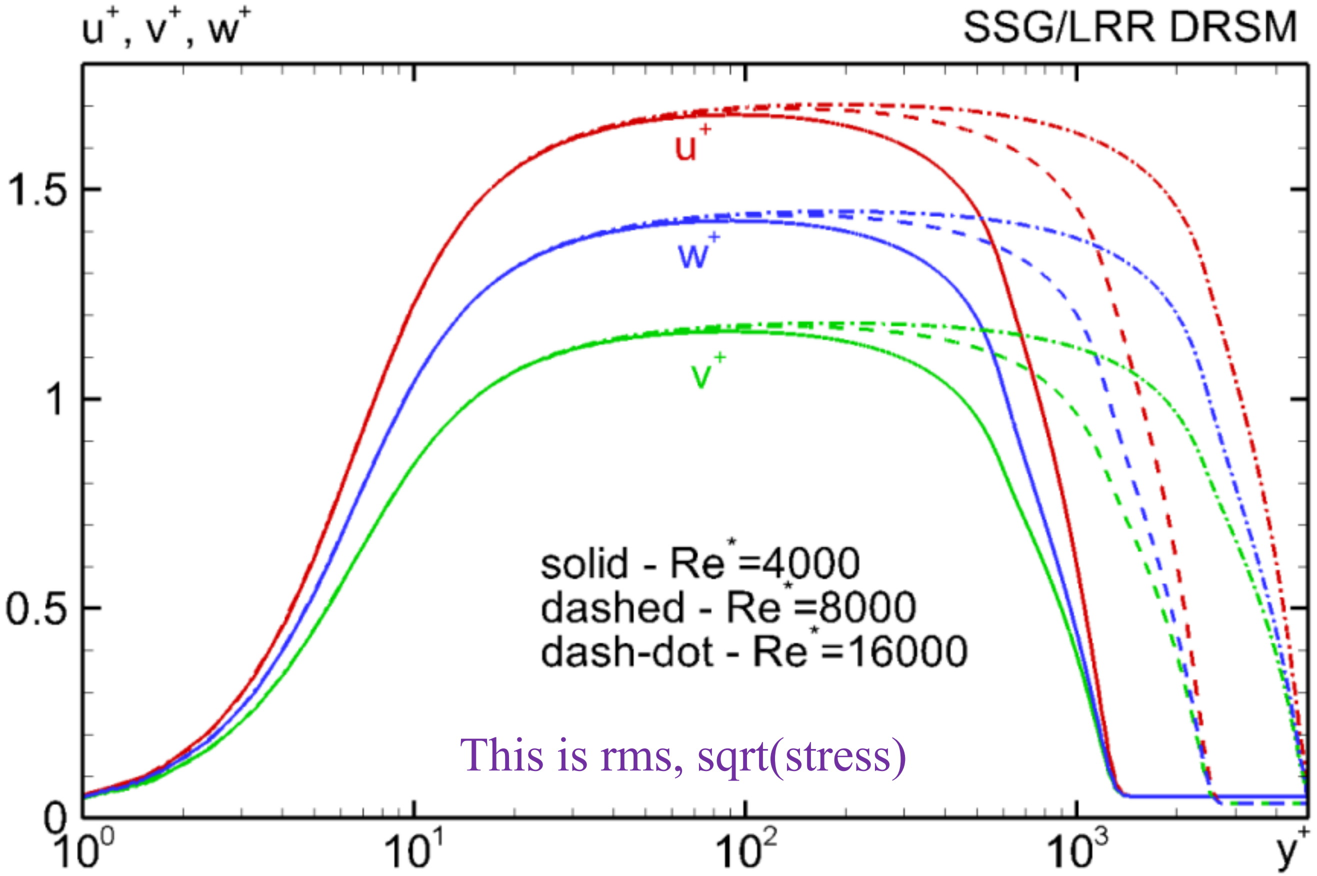
- I discussed this with Prof. Launder…
- The results contradict both GLW implications for the stresses:
  - constant behavior in the log layer, due to $\alpha = 2, \beta = 0$
  - lack of dependence on the flow Reynolds number ("$f(y^+)$")
- Recall peak values, $\overline{u'^2}^+ \approx 7.3,\ k^+ \approx 5.2$

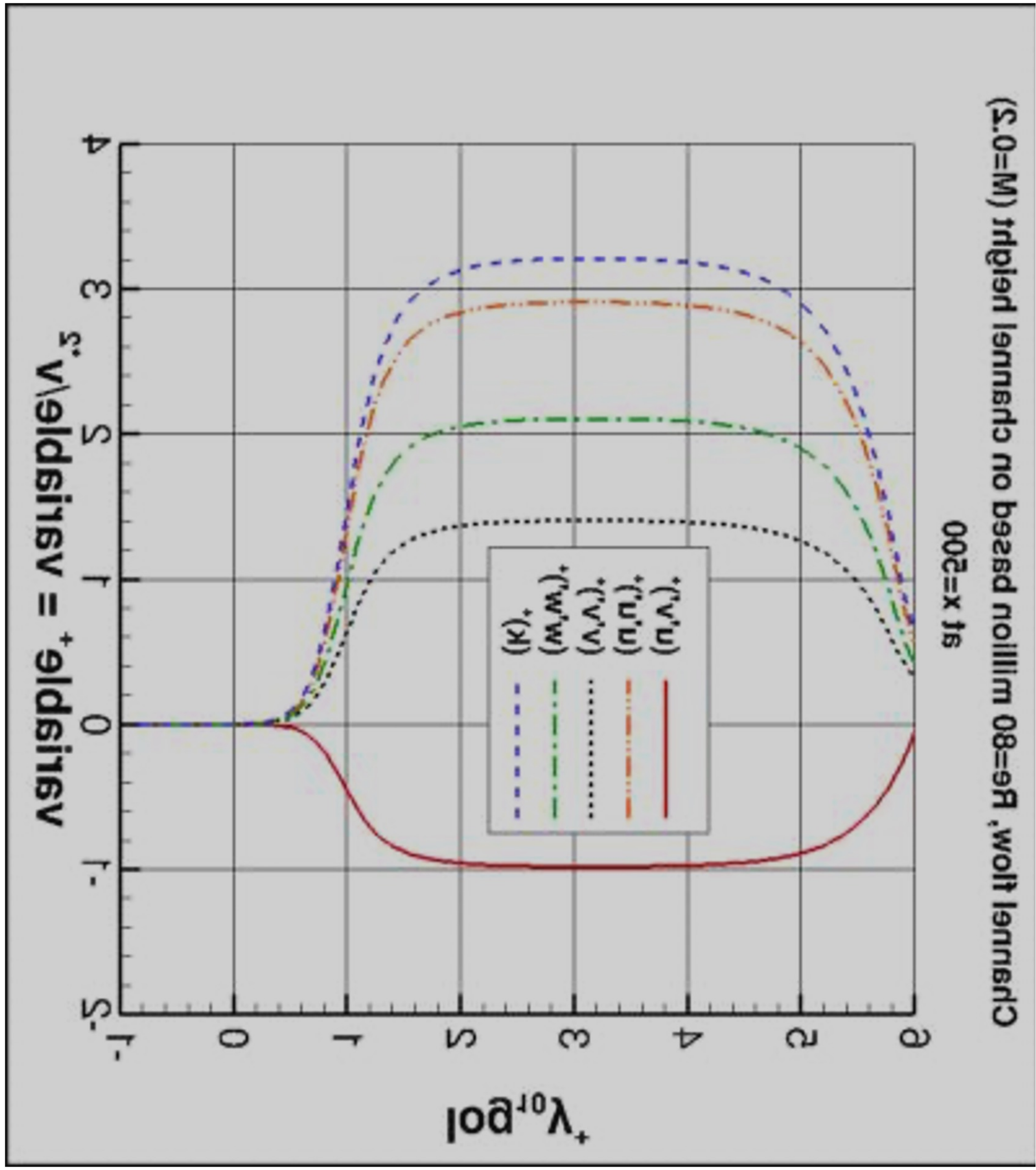




SSG-LRR Reynolds-Stress Model

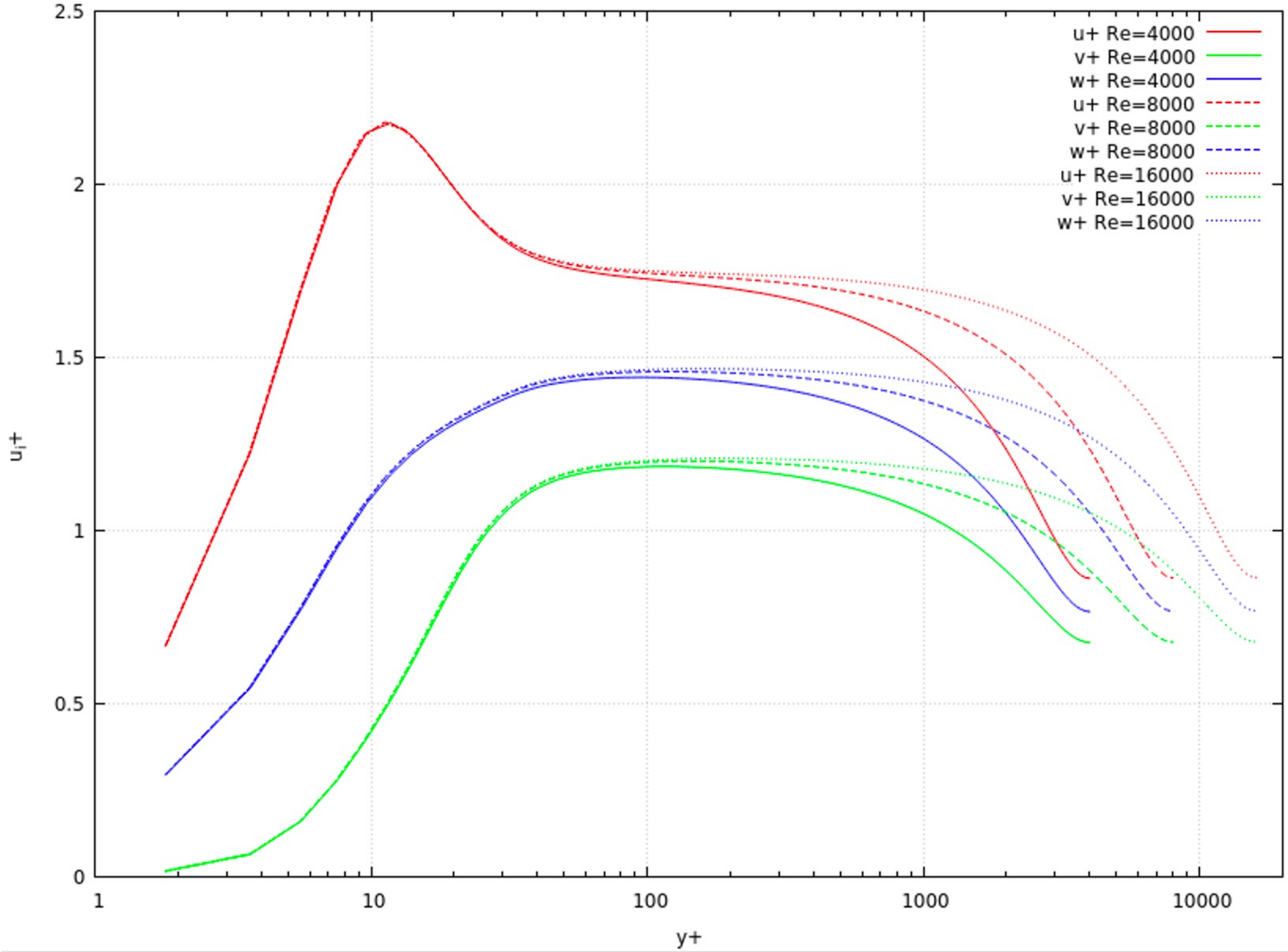
- Courtesy: B. Eisfeld and C. Rumsey
- This model is closest to first principles, but is Classical! $\overline{u_i' u_j'} = f_{ij}(y^+)\, C_{ij}\, u_\tau^2\, y^0$





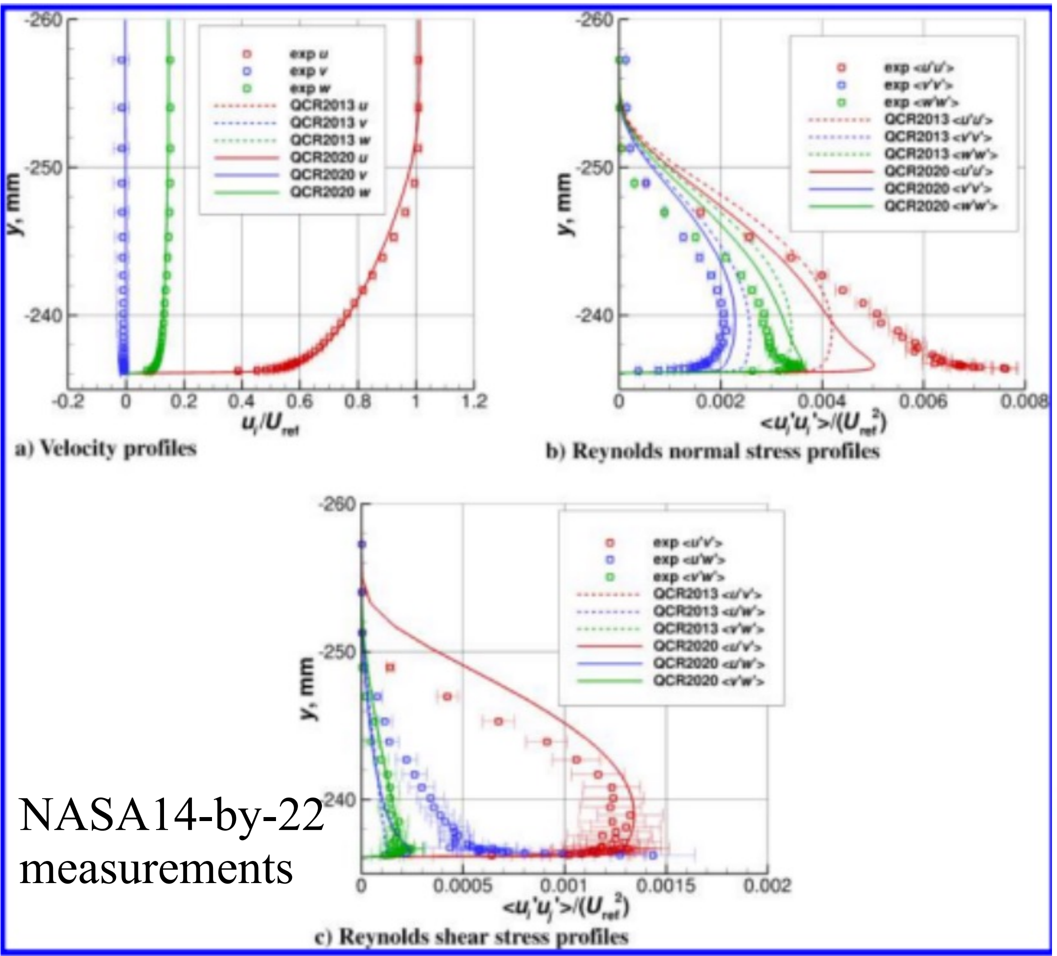
Modified Craft-Launder Reynolds-Stress Model

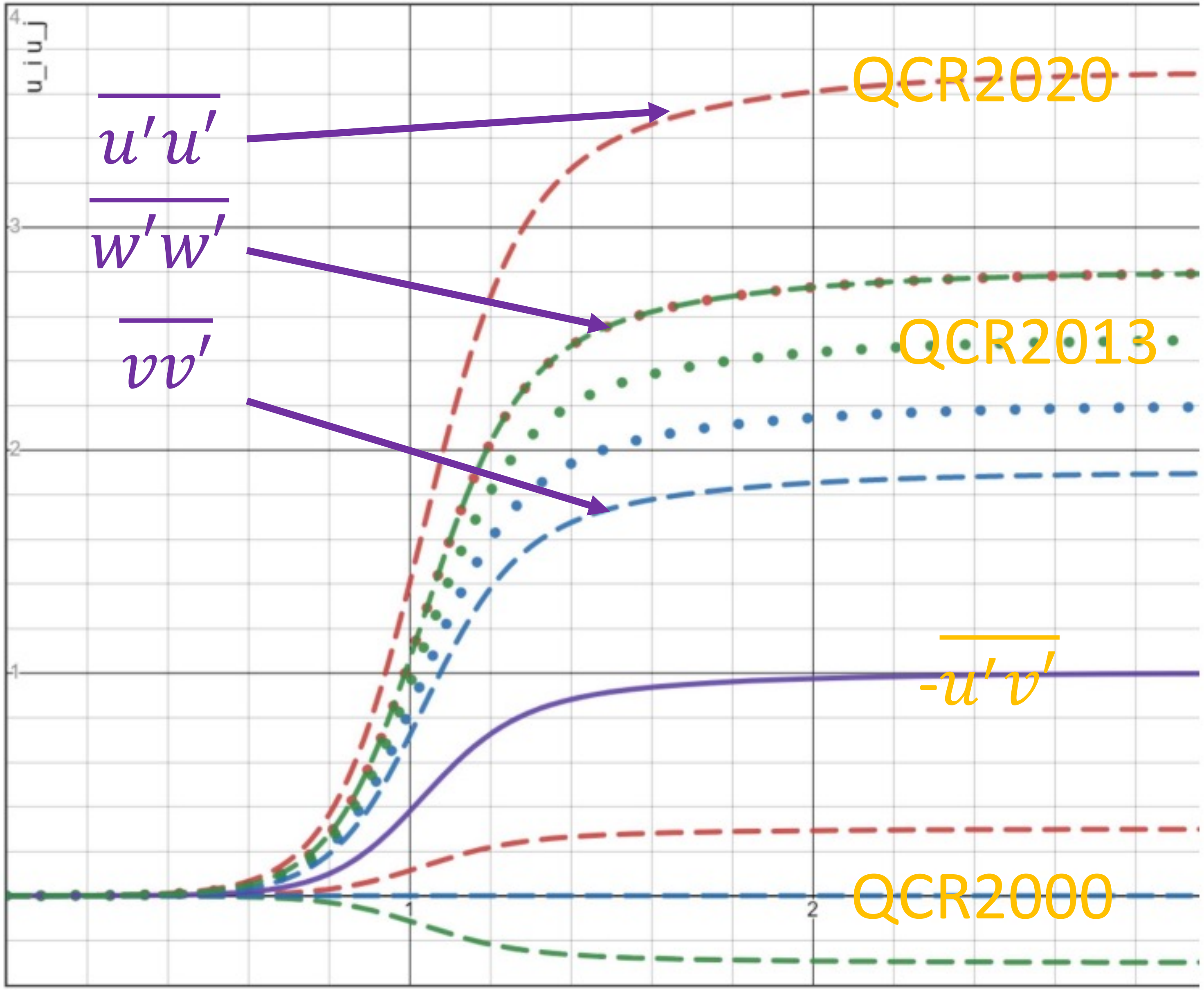
- Courtesy: P. Batten with CFD++
- Channel flow
- Also exhibits GLW
- Peak $\overline{u'^2}^+ \approx 4.8$



QCR Models

- Also exhibit GLW
- $\overline{u_i' u_j'} = g_{ij}\, \overline{u' v'}$
- Anisotropy is fixed
- Models did improve!
- Still missing the near-wall peak, $\overline{u'^2}^{+} \approx 6!$





Does this all Matter?

- Turbulent CFD has lived with this "problem" since the 1970's
  - Its principal arena of success is the boundary layer, and $\overline{u'v'}$ dominates this flow
  - The stresses that fail the GLW enter $\partial/\partial x$ and $\partial/\partial z$, which are $\ll \partial/\partial y$
  - The SA92 model is not even realizable!
- Recent modeling work has brought out the other stresses, for corner flows
  - QCR and similar nonlinear (but Classical) eddy-viscosity models help
  - They still predict the GLW and miss the inner peaks, see QCR2020
- Machine Learning is taking place within at least two styles:
  - Using detailed quantities from DNS, e.g. HiFi-TURB
    - This is more "scientific," and more vulnerable to the Limitation
  - Using only outputs such as lift and pressure
    - This is immune to the Structural Limitation, but it could be superficial
- If training a Classical Model to the Reynolds stresses in channel DNS, Machine Learning will attempt to break through the Limitation by using the viscous functions outside the viscous region
- Could the GLW apply at "enormous" Reynolds numbers?
- Theories of "inactive motion" and the Attached-Eddy Hypothesis explain both failures of the GLW, but they do not connect with Classical Models
- Are there "non-classical" but CFD-friendly models waiting to be created?